Notes about Euler equations

Derivation of the Euler equations can be found in Rezzolla and Zanotti (2013):

$$\partial_t \rho + \partial_x(\rho v) = 0,$$

$$\partial_t v + v\partial_x v + \frac{1}{\rho}\partial_x p = 0,$$

$$\rho\partial_t \epsilon + \rho v \partial_x \epsilon + p\partial_x = 0,$$

where the first equation expresses conservation of mass, the second equation is the actual Euler equation (and all three equations are usually called hydrodynamics equations) and expresses conservation of linear momentum and the first one describes conservation of internal energy. The variables appearing here are the energy density ρ, the velocity v, the internal energy density ϵ and the pressure p. By introducing the (Newtonian) total energy density

$$e = \rho\epsilon + \frac{1}{2}\rho v^2$$

one can rewrite the last equation in the form

$$\partial_t(\frac{1}{2}\rho v^2 + \rho\epsilon) + \partial_x((\frac{1}{2}\rho v^2 + \rho\epsilon + p)v) = 0.$$

As one can see the hydrodynamics equations are a set of three PDEs, but four independent variables appear. Because of this, it is required to specify an additional equation that relates the quantities ρ, v, e, p, e.g. an equation of state

$$p = p(\rho, v, e).$$

Hydrodynamics equations Guermond and Pasquetti (2008)

Euler equations of gas dynamics in conservation form and in 1D reads

$$\partial_t \vec{u} + \partial_x \vec{f}(\vec{u}) = 0,$$

$$\vec{u} = \begin{bmatrix} \rho \\ q \\ E \end{bmatrix}, \qquad \vec{f}(\vec{u}) = \begin{bmatrix} q \\ qv + p \\ v(E+p) \end{bmatrix},$$

where ρ is the density of the gas, v is the velocity, $q = \rho v$ is the momentum, E is the total energy per unit volume and $p = (\gamma - 1)(E - \rho v^2/2)$ is the pressure, $\gamma = 7/5$ for a perfect gas.

Test problems

Smooth problem

The following problem was taken over from Qiu and Shu (2005), Guo and Liu (2009) and adapted to my needs: The initial data

$$\rho(0,x) = 1 + \frac{1}{5}sin(2\pi x),$$

$$q(0,x) = \rho(0,x)$$

$$E(0,x) = \frac{1}{2}q(0,x),$$

is given on the periodic interval $x \in [0,1)$. The corresponding analytic solution to this problem is

$$\rho(t,x) = 1 + \frac{1}{5}sin(2\pi(x-t)),$$

$$q(t,x) = \rho(t,x)$$

$$E(t,x) = \frac{1}{2}q(t,x).$$

Non smooth problems / Riemann problems

Guermond and Pasquetti (2008) gives a list of three test problems for the 1D case. Because I had problems reproducing results for all these cases I also looked at other test problems. The paper Liska and Wendroff (2003) gives a set containing eight problems (one of these uses a different gas constant, another one is specified on a different domain). These test problems should serve as a check for my DG code to see if it works properly.

Guermond, Jean-Luc, and Richard Pasquetti. 2008. *Entropy-Based Nonlinear Viscosity for Fourier Approximations of Conservation Laws*. *Comptes Rendus Mathematique*. Vol. 346. 13-14. Elsevier.

Guo, Yan, and Ru-xun Liu. 2009. "Characteristic-Based Finite Volume Scheme for 1D Euler Equations." *Applied Mathematics and Mechanics* 30 (3): 303–12.

Liska, Richard, and Burton Wendroff. 2003. "Comparison of Several Difference Schemes on 1D and 2D Test Problems for the Euler Equations." *SIAM Journal on Scientific Computing* 25 (3): 995–1017.

Qiu, Jianxian, and Chi-Wang Shu. 2005. "Hermite Weno Schemes and Their Application as Limiters for Runge–Kutta Discontinuous Galerkin Method Ii: Two Dimensional Case." *Computers & Fluids* 34 (6): 642–63.

Rezzolla, Luciano, and Olindo Zanotti. 2013. *Relativistic Hydrodynamics*. Oxford University Press.