Notes wave equation in conservation form

I study the wave equation in conservation form. I do this to check whether I have understood and implemented DG methods correctly.

Wave equation as conservation laws

The conservation form of the wave equation is given in LeVeque (1992).

The wave equation in 1D reads

$$u_{tt} = c^2 u_{xx},$$

where c is the wave speed, which I set to $c = 1$ for the remainder.

Introduction of the variables

$$v = u_x, \qquad w = u_t$$

allows to rewrite the wave equation into a coupled system of first order PDEs. Using commutation of partial derivatives, e.g. $v_t = w_x$ then gives

$$v_t - w_x = 0, \qquad w_t - v_x = 0.$$

Initial conditions for the wave equation

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x),$$

transform to initial conditions for v, w according to

$$v(x,0) = u_0'(x), \qquad w(x,0) = u_1(x).$$

Implementation

In the implementation I use a Lax-Friedrich flux as the numerical flux for v, w with local wave speed given by $c = 1$.

To check my implementation I evolve the initial data

$$u_0(x) = ae^{-b(x-x_0)^2}, \qquad u_1(x) = 2ab(x-x_0)e^{-b(x-x_0)^2},$$

which has the following analytic solution

$$u(t,x) = ae^{-b((x-x_0)-t)^2}.$$

How to recover u from v, w

This can be done either by a line integral or by solving the equation $u_t = w$. I chose the latter approach, because I am already working with an ODE solver in my DG program.

LeVeque, Randall J. 1992. *Numerical Methods for Conservation Laws.* Vol. 3. Springer.