Time asymmetric initial data for critical collapse

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1 The problem

We want to study [3, section 2.2] which investigates the Abraham and Evans papers and challenges their results. In particular, the problem at hand is finding the solution to the constraint equations of General Relativity for a "time asymmmetric" initial slice of spacetime. The following assumptions are taken directly from [3], which in turn emphasize that its their interpretation of what was supposedly worked out in the Abrahams and Evans papers. That is, in spherical coordinates (r, θ, φ) the induced metric is assumed to be conformally flat, whereas the extrinsic curvature should be maximally sliced and non-rotating, e.g.

$$(\gamma_{ij}) = \psi^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin(\theta)^2 \end{bmatrix}, \qquad (K^i_j) = \begin{bmatrix} K^r_r & K^r_\theta & 0 \\ K^\theta_r & K^\theta_\theta & 0 \\ 0 & 0 & K^\varphi_\varphi \end{bmatrix}, \tag{1}$$

where maximal slicing implies $K := K^i_i = K^r_r + K^\theta_\theta + K^\varphi_\varphi = 0$. Furthermore, the component K^r_θ is assumed to be given function of r, θ.

The Hamiltonian and Momentum constraints in vacuum read [1, section 3]

$$\mathcal{H} = R + K^2 + K^{ij}K_{ij} = 0, \tag{2}$$

$$\mathcal{M} = D_j(K^{ij} - \gamma^{ij}K) = 0, \tag{3}$$

where R is the Ricci scalar associated with γ_{ij} and D_i is a covariant derivative that is compatible with γ_{ij}. Plugging in the above Ansätze into these equations yields the following system of PDEs for the unknowns $\psi, K^r_r, K^\theta_\theta$ (taken verbatim from [3])

$$0 = -\frac{8}{\psi^5}\Delta\psi + K^i_j K^j_i, \tag{4}$$

$$0 = \partial_r K^r_r + \frac{6K^r_r\partial_r\psi}{\psi} + \frac{6K^r_\theta\partial_\theta\psi}{r^2\psi} + \frac{3}{r}K^r_r + \frac{1}{r^2}\partial_\theta K^r_\theta + \frac{1}{r^2\tan(\theta)}K^r_\theta, \tag{5}$$

$$0 = -\partial_\theta K^r_r - \partial_\theta K^\varphi_\varphi - \frac{6(K^r_r + K^\varphi_\varphi)\partial_\theta\psi}{\psi} - \frac{K^r_r + 2K^\varphi_\varphi}{\tan(\theta)} + \frac{6K^r_\theta\partial_r\psi}{\psi} + \frac{2K^r_\theta}{r}. \tag{6}$$

Does this version of the equations contain the missing factor Daniela discovered?

After stating the equations [3] proceeds to discuss the choice for K^r_θ they think Abrahams and Evans used in their work as well as their own version. I think in one of the later chapters he talks in more detail about the numerical implementation, which I think just uses a standard pseudospectral method an a Newton-Raphson scheme to solve the resulting non-linear algebraic equations.

After all, it appears that the problem is already solved. So why bother? Well, Daniela might wanna study other variations of this kind of initial data in the future and so it could certainly be of no harm if one understands the problem a bit better. Furthermore, Khrinov did not succeed in reproducing Abrahams' and Evan's data so there remains the question of whether they overlooked something or not.

A few observations follow:

1. All three equations are coupled.
2. K^r_θ takes the role of a source term in all equations.
3. What boundary conditions should be applied? (Probably asympotic flatness and regularity around the coordinate origin as well as regularity along the symmetry axis and reflection symmetri across an equatorial plane.)
4. The PDEs are first order in K^r_r, K^θ_θ and second order in ψ.

2 Attempted solution

2.1 Initial idea for alternative approach

The initial idea for this came when trying to rewrite the above equations using the methods presented in [1, section 3.1.3] on conformal transformations of the extrinsic curvature (keyword vector potential). After playing around for a bit I realized that the *source* term K^r_θ are somehow in the way. So I was wondering whether one could proceed an approach similiar to what helped me with solving the Rotating Mass Shell (RMS) problem, that is, solve a different but simpler problem first and then use the *source* terms to construct the desired solution. This kind of approach to problem solving is not really new, but instead often goes under the phrase *picking the right coordinate system for the right problem* (source: myself).

The starting point for the RMS problem was to realized that axisymmetric stationary vacuum spacetimes are characterized exactly by only three metric potentials in Weyl-Lewis-Papapetrou coordinates, whereas I have always worked with four. As it turned out (although I knew this for quiet some time) the superficial fourth potential is just the conformal factor that relates these Weyl coordinates with the quasi-isotropic coordinates the RMS problem was posed with.

The problem at hand is similar but also quite different from the RMS problem. Firstly, the RMS problem aimed to construct a solution to the full Einstein field equations (EFEs) for a stationary spacetime. That is, the resulting potentials also include information on a particular slicing. Whereas here, we are only concerned with the constraint equations without making any reference to how that initial slice will later be embedded into a time evolution. The two problems have in common that they both are concerned with vacuum spacetimes (the matter part in the RMS problem is confined to an infinitely thin shell) and axisymmetry. One might naively say that the problem at hand appears to be simpler, because it does not involve rotation, but that is probably not true.

2.2 Simplifying the Ansatz

Arguably, a simpler problem to solve would be the one where $K^r_\theta = 0$. In this case the Ansatz for K^i_j would become diagonal which in turn would simplify a vector potential formulation of K^{ij}. Assuming $K^r_\theta = 0$, however, is not of much help unless we can show that a problem where $K^{i'j'}$ is diagonal is equivalent to the initial problem and can be related through a coordinate transformation.

As a starting point we recall a theorem about eigen systems that often finds application in quantum mechanics class (and I think also when studying (quasi-)normal oscillations in mechanics, not 100% sure though): A set of diagonalizable matrices commutes if and only if the set of matrices is simultaneously diagonalizable [4, 2]. Given that γ^{ij}

is already diagonal, hence, also commutes with K^{ij}, it is tempting to apply the above theorem and try to find a new coordinate system in which both are diagonal.

We will now switch up notation. Let $(x^{i'}) = (r', \theta', \varphi')$ denote the coordinates we introduced in section 1 and let's put primes on all indices of all components used before. Furthermore, we switch coordinates from spherical to cylindrical $(x^{i'}) = (\rho', z', \varphi')$ and rewrite (1) as

$$(\gamma_{i'j'}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho'^2 \end{bmatrix}, \qquad (K^{i'j'}) = \begin{bmatrix} K^{\rho'\rho'} & K^{\rho'z'} & 0 \\ K^{z'\rho'} & K^{z'z'} & 0 \\ 0 & 0 & K^{\varphi'\varphi'} \end{bmatrix}. \tag{7}$$

We note that $K^{z'\rho'} = K^{\rho'z'}$, whereas $K^{z'}_{\rho'} \neq K^{\rho'}_{z'}$ in general. The reason for using cylindrical over spherical coordinates here is that it is simpler to deal with the transformations later. Furthermore, this does not incure any loss of generality, because the transformation between those two systems is known analytically and does not require solving any additional equations.

Next, we diagonalize $K^{i'j'}$. To this end we compute its characteristic polynomial

$$\det\left(K^{i'j'} - \lambda\delta^{i'j'}\right) = ((K^{\rho'\rho'} - \lambda)(K^{z'z'} - \lambda) - K^{\rho'z'2})(K^{\varphi'\varphi'} - \lambda) \tag{8}$$

The eigenvalues are the zeros of the above polynomial and read

$$\boxed{K_\pm = \frac{1}{2}(K^{\rho'\rho'} + K^{z'z'}) \pm \frac{1}{2}\sqrt{(K^{\rho'\rho'} - K^{z'z'})^2 + 4K^{\rho'z'2}}, \qquad K_0 = K^{\varphi'\varphi'}.} \tag{9}$$

The corresponding (normalized) eigenvectors read

$$\boxed{v_\pm = N_\pm \begin{bmatrix} K_\pm - K^{z'z'} \\ K^{\rho'z'} \\ 0 \end{bmatrix}, \qquad v_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},} \tag{10}$$

where $N_\pm = 1/\sqrt{K_\pm^2 + K^{zz2} + K^{\rho z2} - 2K^{zz}K_\pm}$. TODO Not 100% sure if eigenvectors are correct, because `sympy` did not simplify them nicely and I am too lazy to verify by hand, plus they are irrelevant at the moment.

Because $K^{i'j'}$ is real symmetric, the matrix

$$J = (J^{i'}_{j}) = \begin{bmatrix} v_+ & v_- & v_0 \end{bmatrix} \tag{11}$$

is orthogonal $J^T = J^{-1}$ and we can use it to construct the desired diagonal matrix

$$\boxed{(K^{ij}) = ((J^{-1})^{i}_{i'} K^{i'j'} J^{j}_{j'}) = \begin{bmatrix} K_+ & 0 & 0 \\ 0 & K_- & 0 \\ 0 & 0 & K_0 \end{bmatrix}.} \tag{12}$$

This last equations suggests now that the diagonalizer $J^i_{i'}$ might be adopted as the Jacobi matrix corresponding to a coordinate transformation of the form

$$\rho = \rho(\rho', z'), \qquad\qquad z = z(\rho', z'), \tag{13}$$

such that

$$(\mathrm{d}x^i) = \begin{bmatrix}\mathrm{d}\rho \\ \mathrm{d}z\end{bmatrix} = \begin{bmatrix}\rho_{\rho'} & \rho_{z'} \\ z_{\rho'} & z_{z'}\end{bmatrix} \cdot \begin{bmatrix}\mathrm{d}\rho' \\ \mathrm{d}z'\end{bmatrix} = ((J^{-1})^i_{i'}\mathrm{d}x^{i'}), \tag{14}$$

$$(\mathrm{d}x^{i'}) = \begin{bmatrix}\mathrm{d}\rho' \\ \mathrm{d}z'\end{bmatrix} = \begin{bmatrix}\rho'_\rho & \rho'_z \\ z'_\rho & z'_z\end{bmatrix} \cdot \begin{bmatrix}\mathrm{d}\rho \\ \mathrm{d}z\end{bmatrix} = (J^i_{i'}\mathrm{d}x^i). \tag{15}$$

Because J is orthogonal and block diagonal, one might be tempted to rewrite it as

$$(J^i_{i'}) = \begin{bmatrix}\cos(\chi) & \sin(\chi) & 0 \\ -\sin(\chi) & \cos(\chi) & 0 \\ 0 & 0 & 1\end{bmatrix}, \tag{16}$$

with

$$\cos(\chi) = \frac{K_+}{N_+} = \frac{2K^{\rho'z'}}{N_-}, \qquad\qquad \sin(\chi) = \frac{K_-}{N_-} = -\frac{2K^{\rho'z'}}{N_+}, \tag{17}$$

which emphasizes the orthogonality. However, this is too restrictive for our case. Indeed, the following leads to a contradiction if $\chi \neq \mathit{const.}$:

$$\rho'_{\rho z} = -\sin(\chi)\chi_{,z}, \quad \rho'_{z\rho} = \cos(\chi)\chi_{,\rho}, \quad \Rightarrow \quad \chi_{,z} = -\frac{\cos(\chi)}{\sin(\chi)}\chi_{,\rho} = -\cot(\chi)\chi_{,\rho}, \tag{18}$$

$$z'_{\rho z} = -\cos(\chi)\chi_{,z}, \quad z'_{z\rho} = -\sin(\chi)\chi_{,\rho}, \quad \Rightarrow \quad \chi_{,z} = \frac{\sin(\chi)}{\cos(\chi)}\chi_{,\rho} = \tan(\chi)\chi_{,\rho}. \tag{19}$$

Hence, Ansatz (16) is not compatible with (15), because demanding that $\chi = \mathit{const.}$ implies that (11) is constant which in turn implies that $K^{i'j'}$ is constant in space, which seems too restrictive.

One way to resolve this issue is to introduce a new unknown scale factor $\sigma(\rho, z)$ and replace (16) with

$$(J^i_{i'}) = \sigma\begin{bmatrix}\cos(\chi) & \sin(\chi) & 0 \\ -\sin(\chi) & \cos(\chi) & 0 \\ 0 & 0 & 1\end{bmatrix}. \tag{20}$$

The addition of σ now converts the transformations $\rho(\rho', z'), z(\rho', z')$ into a pair of harmonic functions. That is, they can now be made to satisfy the integrability conditions

$$\rho'_{,\rho z} = \rho_{,z\rho}, \qquad\qquad z'_{,\rho z} = z'_{,z\rho}, \tag{21}$$

as well as the Cauchy-Riemann equations

$$\rho'_{,\rho} = z'_{,z}, \qquad \rho_{,z'} = -z'_{,\rho}, \tag{22}$$

and consequently must satisfy

$$\boxed{\Delta_{cart}\rho' = \rho'_{,\rho\rho} + \rho'_{,zz} = 0, \qquad \Delta_{cart} z' = z'_{,\rho\rho} + z'_{,zz} = 0.} \tag{23}$$

Let us emphasize the occurence of the 2D cartesian Laplace operator Δ_{cart} in the last equation, which is distinct from the 3D cylindrical Laplace operator in axisymmetry $\Delta_{cyl} = \partial_{\rho\rho} + 1/\rho\, \partial_\rho + \partial_{zz}$. From (21) we derive the following relations linking χ and σ,

$$\boxed{\chi_{,\rho} = \frac{\sigma_{,z}}{\sigma}, \qquad \chi_{,z} = -\frac{\sigma_{,\rho}}{\sigma}.} \tag{24}$$

This last relation says that once we gained knowledge about either χ or σ, we can compute the respective other one and then construct the coordinate transformation that allows us to map from ρ, z to ρ', z'.

With the correct Jacobi matrix at hand we can write down the induced metric in the new coordinates,

$$\boxed{\gamma_{ij} = \Psi^4 \bar{\gamma}_{ij} = \Psi^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & V^2 \end{bmatrix}, \quad V = \frac{\rho'(\rho, z)}{\sigma(\rho, z)}, \quad \Psi^4 = \psi(\rho'(\rho, z), z'(\rho, z))^4 \sigma(\rho, z)^2.} \tag{25}$$

What we have shown here is that if we can somehow get a handle on the quantities $K_\pm, K_0, \Psi, V$, we can use the solution to (23) to obtain σ and then χ through (24) and ultimately compute $K^{\rho'\rho'}, K^{z'z'}, K^{\varphi'\varphi'}$ using a *source* function $K^{\rho' z'}$ re-parameterized by ρ, z. To be honest, now written out explicitly this sounds more involved than solving the initial problem. But perhaps the solution of the subproblems in this approach are simpler then solving the PDEs (4)-(6) all at once.

2.3 Constraint equations with new Ansatz

We now write out the constraint equations (3) using Ansätze (12), (25) and the *tricks* provided in [1, section 3.1.3].

We start by noting that the assumption on maximal slicing $K = 0$ is a coordinate invariant statement. In our adapted coordinates it reads

$$\boxed{K = K_+ + K_- + K_0 = 0.} \tag{26}$$

Consequently, we can ignore the trace part in the decomposition of K^{ij} in [1] and just write

$$K^{ij} = A^{ij}. \tag{27}$$

The book then proceeds by applying a conformal transformation to A^{ij} to then find

$$A^{ij} = \Psi^{-10}\bar{A}^{ij}, \qquad D_j A^{ij} = \bar{D}_j \bar{A}^{ij}, \tag{28}$$

where $\bar{D}_j$ is the covariant derivative associated with the conformal metric $\bar{\gamma}_{ij}$. Eventually, one arrives at the following rewriting of (3)

$$8\bar{D}^2\Psi - \Psi\bar{R} + \Psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = 0, \tag{29}$$

$$\bar{D}_j\bar{A}^{ij} = 0. \tag{30}$$

We emphasize that this form of the constraint equations with Ansätze (12), (25) differs by an additional factor $\bar{R}$ which would have been absent from when casting the Ansätze (1) into this form, because there the induced metric is conformally flat. We also note that (30) is now coupled to (29) through the unknown V appearing in the conformal covariant derivative, whereas in the initial formulation of the problem they were decoupled, because of conformal flatness. Not sure if the last statement is correct, because also in the initial formulation of the problem we have $K^{ij} = A^{ij} = \psi^{-10}\bar{A}^{ij}$, but the PDE systems shows a coupling between the equations for ψ and K^r_r, K^θ_θ. I remember looking into this already (also using the vector potential formulation) and concluded that this is not really simpler. Maybe I should check again?

Next [1] discusses how $\bar{A}^{ij}$ can be further split,

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{A}^{ij}_{L}, \tag{31}$$

e.g. into a transverse-traceless part $\bar{A}^{ij}_{TT}$ that satsifies

$$\bar{D}_j\bar{A}^{ij}_{TT} = 0, \tag{32}$$

as well as a symmetric, traceless gradient of a vector W^i, called the longitudinal part

$$\bar{A}^{ij}_L = \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k =: (\bar{L}W)^{ij}, \tag{33}$$

such that

$$\bar{D}_j \bar{A}_L^{ij} = \bar{D}_j(\bar{L}W)^{ij} = \bar{D}^2 W^i + \frac{1}{3}\bar{D}^i(\bar{D}_j W^j) + \bar{R}^i_j W^j =: (\bar{\Delta}_L W)^{ij}. \tag{34}$$

Plugging this back into (30) we find

$$\boxed{(\bar{\Delta}_L W)^{ij} = 0.} \tag{35}$$

Is this enough to conclude that we only need to focus on the longitudinal part?

Let us investigate how the components of (35) look like. To this end we compute the non-vanishing Christoffel symbols $\bar{\Gamma}^i_{jk}$, the conformal Ricci tensor $\bar{R}^i_j$ as well as the Ricci scalar $\bar{R}$ using my `sympy` notebook and we find

$$\bar{\Gamma}^\rho_{\varphi\varphi} = -VV_{,\rho}, \qquad \bar{\Gamma}^z_{\varphi\varphi} = -VV_{,z}, \qquad \bar{\Gamma}^\varphi_{\rho\varphi} = -\frac{V_{,\rho}}{V}, \qquad \bar{\Gamma}^\varphi_{z\varphi} = -\frac{V_{,z}}{V}, \tag{36}$$

$$\bar{R}^\rho_\rho = -\frac{V_{,\rho\rho}}{V}, \qquad \bar{R}^z_z = -\frac{V_{,zz}}{V}, \qquad \bar{R}^\rho_z = -\frac{V_{,\rho z}}{V}, \qquad \bar{R}^\varphi_\varphi = -\frac{\Delta_{cart} V}{V}, \tag{37}$$

$$\bar{R} = -2\frac{\Delta_{cart} V}{V}. \tag{38}$$

We then evaluate (33) to find

$$\bar{A}^{\rho\rho} = \frac{2}{3}\frac{2VW^\rho_{,\rho} - VW^z_{,z} - V_{,\rho}W^\rho - V_{,z}W^z}{V}, \tag{39}$$

$$\bar{A}^{zz} = \frac{2}{3}\frac{-VW^\rho_{,\rho} + 2VW^z_{,z} - V_{,\rho}W^\rho - V_{,z}W^z}{V}, \tag{40}$$

$$\bar{A}^{\varphi\varphi} = \frac{2}{3}\frac{-V(W^\rho_{,\rho} + W^z_{,z}) + 2V_{,\rho}W^\rho + 2V_{,z}W}{V^3}, \tag{41}$$

$$\bar{A}^{\rho z} = W^\rho_{,z} + W^z_{,\rho}, \qquad \bar{A}^{\rho\varphi} = W^\varphi_{,\rho}, \qquad \bar{A}^{z\varphi} = W^\varphi_{,z}. \tag{42}$$

Because of (12) and (27) $\bar{A}^{ij}$ is also diagonal,

$$\boxed{(\bar{A}^{ij}) = \begin{bmatrix} \bar{A}^{\rho\rho} & 0 & 0 \\ 0 & \bar{A}^{zz} & 0 \\ 0 & 0 & \bar{A}^{\varphi\varphi} \end{bmatrix},} \tag{43}$$

which allows us to conclude that $W^\varphi = const.$ in this Ansatz, so w.l.o.g we set it to zero,

$$\boxed{(W^i) = \begin{bmatrix} W^\rho \\ W^z \\ 0 \end{bmatrix}.} \tag{44}$$

Furthermore, we obtain the identity

$$\boxed{W^{\rho}_{,z} = -W^{z}_{,\rho}.} \tag{45}$$

Direct evaluation of (35) yields

$$0 = 3(\bar{\Delta}W)^{\rho} = 4W^{\rho}_{,\rho\rho} + 2W^{\rho}_{,zz} - 2\frac{V_{,\rho\rho}}{V}W^{\rho} - 2\frac{V_{,\rho z}}{V}W^{z} + 4\frac{V_{,\rho}}{V}W^{\rho}_{,\rho} + 2\frac{V_{,z}}{V}W^{\rho}_{,z} - 4\frac{V_{,\rho}^2}{V^2}W^{\rho} - 4\frac{V_{,\rho}V_{,z}}{V^2}W^{z}, \tag{46}$$

$$0 = 3(\bar{\Delta}W)^{z} = 2W^{z}_{,\rho\rho} + 4W^{z}_{,zz} - 2\frac{V_{,\rho z}}{V}W^{\rho} - 2\frac{V_{,zz}}{V}W^{z} + 2\frac{V_{,\rho}}{V}W^{z}_{,\rho} + \frac{V_{,z}}{V}W^{z}_{,z} - 4\frac{V_{,z}^2}{V^2}W^{z} - \frac{V_{,\rho}V_{,z}}{V^2}W^{\rho}, \tag{47}$$

$$0 = (\bar{\Delta}W)^{\varphi} = W^{\varphi}_{,\rho\rho} + W^{\varphi}_{,zz} + 3\frac{V_{,\rho}}{V}W^{\varphi}_{,\rho} + 3\frac{V_{,z}}{V}W^{\varphi}_{,z}. \tag{48}$$

The last equation is automatically satisfied because of our choice $W^{\varphi} = 0$ and can be ignored. It remains to solve (46), (47) once we know how to determine V.

TODO

- Massage the W PDEs a bit more by undoing total derivatives.
- Expand the Hamiltonian equation with the new Ansatz.
- How should V be determined?
- Do we need to make reference to the diagonalization procedure to be able to solve for V or σ such that it is then compatible with what we started with?

References

[1] Thomas W Baumgarte and Stuart L Shapiro. *Numerical relativity: solving Einstein's equations on the computer*. Cambridge University Press, 2010.

[2] RA Horn and CR Johnson. *Matrix analysis second edition*. 2013.

[3] Anton Khirnov. "Representation of dynamical black hole spacetimes in numerical simulations". In: (2022).

[4] Wikipedia. *Diagonalizable Matrix – Simultaneous diagonalization*. (accessed August 21, 2023).